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Critical points and intermediate phases on wedges of $\mathbb{Z}^d$

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Abstract. We examine the phase structure of self-avoiding walks, the Ising magnet and bond percolation defined on subsets of $\mathbb{Z}^d$, $d \geq 2$, which have the geometry of wedges. We prove that if the cross sectional area of the wedge diverges with its width, then the high temperature critical point of any of these models defined on the wedge coincides with that of the corresponding model on the full lattice. For the Ising magnet and bond percolation, we show that there is a non-trivial low temperature critical point if the cross sectional area of the wedge diverges logarithmically with its width. Moreover, for bond percolation we show that the low temperature critical point of a logarithmic wedge may be made arbitrarily close to that of the full lattice by taking the coefficient of the logarithm large enough. Corollaries to these theorems include the existence of an intermediate phase for the Ising magnet and bond percolation on logarithmic wedges and the existence of a first-order transition for percolation models on a subclass of these wedges.

1. Introduction and summary of results

Recently there has been some interest in the critical behaviour of statistical mechanical models defined on subsets of the hypercubic lattice $\mathbb{Z}^d$. Cardy (1983) used renormalisation group methods to determine the critical behaviour of $O(N)$ magnets defined on a wedge-like subset of $\mathbb{Z}^d$, bounded by two $(d-1)$-dimensional hyperplanes. Daoud *et al* (1975) and Barber *et al* (1978) considered similar questions for polymers. More recently, there have been studies of self-avoiding walks in wedges by Cardy and Redner (1984) and Guttman and Torrie (1984).

Motivated by this work, Hammersley and Whittington (1985) studied the connectivity constant for self-avoiding walks on wedges of $\mathbb{Z}^d$. They defined a wedge, $\mathbb{Z}^d(f)$, to be a subset of $\mathbb{Z}^d$ with coordinates $(x_1, \ldots, x_d)$ satisfying $x_1 \geq 0, 0 \leq |x_i| \leq f_i(x_1)$ for $i = 2, \ldots, d$. Hammersley and Whittington proved that if $f_i(x) \to \infty$ as $x \to \infty$, then the connectivity constant of the wedge $\mathbb{Z}^d(f)$ is identical to that of the full d-dimensional lattice. This result seemed to be in curious contrast to the work of Grimmett (1981, 1983) on two-dimensional bond percolation. Grimmett proved that a 'wedge' must widen *at least logarithmically* with distance from the origin (with coefficient depending on $p - p_c$) in order that there be percolation in the wedge at density p. This apparent inconsistency between self-avoiding walks and two-dimensional percolation was noted

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by Hammersley and Whittington (1985) who, in addition, showed that a weaker form of the Grimmett result holds for bond percolation in all $d \geqslant 2$ and conjectured that similar results should hold also for the Ising magnet.

In this paper, we examine the types of questions addressed by Grimmett (1981, 1983) and Hammersley and Whittington (1985) in the context of self-avoiding walks, bond percolation and the Ising magnet. The purpose of our analysis is threefold: first, we resolve the apparent inconsistency between the behaviour of self-avoiding walks and percolation models in wedges. Second, we extend several of the results of Grimmett and Hammersley and Whittington to a larger class of models and to general dimension. Finally, by combining some of these results, we show that it is possible to obtain rather unusual phase structures for percolation and the Ising magnet in wedges of certain geometry.

In order to explain the distinction between the percolation results of Grimmett and the self-avoiding walk results of Hammersley and Whittington, we recall some basic properties concerning the phase structure of these models. Since the phase structure of percolation is similar to that of the more familiar Ising magnet, let us begin by reviewing the latter. As usual, the Ising magnet is defined by the partition function

$$Z_\Lambda(\beta) = \sum_{\{\sigma_i\}} \exp[-\beta H_\Lambda(\{\sigma_i\})] \qquad H_\Lambda(\{\sigma_i\}) = -\sum_{\langle i,j\rangle \in \Lambda} \sigma_i \sigma_j \tag{1.1}$$

where the σ_i takes values ± 1 and $\langle i, j\rangle$ denotes a nearest-neighbour pair of $\Lambda \subset \mathbb{Z}^d$. The high temperature ($\beta \ll 1$) phase is characterised by exponential decay of correlations, $\langle \sigma_i \sigma_j \rangle \sim \exp[-M(\beta)|i-j|]$, while (in $d \geqslant 2$) the low temperature ($\beta \gg 1$) phase is characterised by a non-zero spontaneous magnetisation, $\langle \sigma_0 \rangle_{\beta,+} > 0$. (Here $\langle\ \rangle_{\beta,+}$ denotes expectation with respect to the Gibbs measure induced by plus boundary conditions.)

Similar phases occur for the bond percolation model. Here one takes the bonds of $\mathbb{Z}^d$ to be occupied with independent probability $p \in [0, 1]$. Each configuration of occupied bonds divides the lattice into connected components. The analogue of the two-point function is the connectivity, τ_{ij}, which is the probability (with respect to Bernoulli measure) that i is in the connected component, $C(j)$, of occupied bonds of j:

$$\tau_{ij}(p) = \mathrm{Prob}_p\{i \in C(j)\}. \tag{1.2}$$

The existence (with probability one) of an infinite cluster is determined by whether the percolation probability

$$P_\infty(p) = \mathrm{Prob}_p\{|C(0)| = \infty\} \tag{1.3}$$

is positive. For this model, the low density ($p \ll 1$) phase is characterised by exponential decay of connectivities, $\tau_{ij}(p) \sim \exp[-m(p)|i-j|]$, while (in $d \geqslant 2$) the high density ($(1-p) \ll 1$) phase is characterised by percolation, i.e. $P_\infty(p) > 0$.

As should be clear from the above discussion, one can define two *a priori* different critical points for either the Ising magnet or percolation. In the former case, there is a low temperature critical point, β_c^l, such that the system exhibits spontaneous magnetisation for $\beta > \beta_c^l$, and a high temperature critical point, $\beta_c^h \leqslant \beta_c^l$, such that correlations decay exponentially whenever $\beta < \beta_c^h$. Similarly, one may define the percolation threshold, p_c, as the point above which $P_\infty(p) > 0$, and the critical point $\pi_c \leqslant p_c$, below which $\tau_{ij}(p)$ decays exponentially. Such points need not coincide; indeed, as proved by Fröhlich and Spencer (1981), the $\mathbb{Z}_n$ models for n sufficiently large have an intermediate (Kosterlitz-Thouless (1973) type of) phase in which there is algebraic decay of correlations. The absence of an intermediate phase for the Ising magnet on the full lattice has been known for $d=2$ since the exact solution of Onsager (1944) and has been established for $d \geq 3$ by Aizenman (1985). For bond percolation on the hypercubic lattice, Kesten (1980) has shown that $\pi_c = p_c$ in $d=2$, but similar results (though widely believed) have not been proved for $d \geq 3$.

The self-avoiding walk is distinguished from the above models in that a low temperature phase does not occur. This is a consequence of the fact that the self-avoiding walk is defined by a generating function

$$G_{ij}(\alpha) = \sum_{w: i \to j} \exp(-\alpha |w|) \tag{1.4}$$

which is the analogue of the two-point function in the Ising magnet or the connectivity in bond percolation. In (1.4), the sum is over all self-avoiding walks from i to j along the bonds of $\mathbb{Z}^d$ and $|w|$ is the length of the walk w. Viewed in this light, it is clear that self-avoiding walks have *no* phase characterised by a non-zero order parameter analogous to the spontaneous magnetisation or the percolation probability. Indeed, self-avoiding walks have only a 'high temperature' critical point α_c, above which $G_{ij}(\alpha) \sim \exp(-\mu(\alpha)|i-j|)$, and below which these quantities diverge.

The distinction between the type of behaviour found by Hammersley and Whittington (1985) for self-avoiding walks and that found by Grimmett (1981, 1983) for two-dimensional percolation is simply that the former is characteristic of the 'high temperature' critical point, while the latter pertains to the 'low temperature' transition. In order to establish a connection between the high temperature transition and the self-avoiding walk result of Hammersley and Whittington, in § 3 we discuss some entropic considerations which prove that $\alpha_c \equiv \inf\{\alpha | \mu(\alpha) > 0\}$ is precisely the logarithm of the connectivity. Thus the Hammersley and Whittington result may be reformulated by saying that whenever self-avoiding walks are defined on a wedge of divergent width, the 'high temperature' critical point coincides with that of the full lattice. In § 3, we prove that such a result holds not only for self-avoiding walks, but also for the Ising magnet and bond percolation. Indeed, coincidence of the high temperature critical point of any divergent wedge with that of the full lattice is generic.

In § 4, we turn to a consideration of the low temperature phase, and thus restrict attention to the Ising magnet and percolation. Here, the issue is the rate at which the wedge must widen in order to obtain either spontaneous magnetisation or percolation. Hammersley and Whittington (1985) proved that for bond percolation on $\mathbb{Z}^d$, $d \geq 2$, the necessary and sufficient condition to achieve percolation for p sufficiently close to one is that a (symmetric) wedge widen at least as fast as the $(1/(d-1))$th root of the logarithm of the distance from the origin. (In other words, the cross sectional area must grow at least as fast as the logarithm of the distance.) In § 4, we provide a proof of this type of behaviour which holds for *both* percolation (for p sufficiently close to one) and the Ising magnet (for β sufficiently large). Our proof implies that there exist wedges with logarithmically growing cross sectional area such that $p_c < p_c(W) < 1$ and $\beta_c^l < \beta_c^l(W) < \infty$, where $p_c(W)$ and $\beta_c^l(W)$ denote the percolation threshold and low temperature critical point, respectively, of the wedge. Combining this with high temperature results of § 3—namely, that $\pi_c = \pi_c(W)$ and $\beta_c^h = \beta_c^h(W)$—it is clear that such systems exhibit an intermediate phase. This phase is characterised by subexponential decay of correlations, but zero spontaneous magnetisation or percolation probability.

In § 5, we address more sensitive questions concerning the low temperature (or high density) critical point in percolation: namely, if $p > p_c$, how fast must a wedge widen in order that it exhibit percolation at density p? This is precisely the question answered by Grimmett (1981, 1983) for two-dimensional percolation. Grimmett showed that whenever p exceeds the two-dimensional percolation threshold, there will be percolation in a logarithmically growing wedge on $\mathbb{Z}^2$ provided that the coefficient of the logarithm is sufficiently large. Hammersley and Whittington (1985) conjectured that analogous results should hold for $d \geqslant 3$. Partial, but not optimal, results along these lines have been obtained by Grimmett (1984) who showed that it is possible to achieve percolation in a wedge of $\mathbb{Z}^3$ of logarithmic cross section if p exceeds $\frac{1}{2}$ (the *two-dimensional* threshold). In § 5, we use rescaling techniques to prove the Hammersley and Whittington (1985). In order to avoid excessive provisos, we restrict the (appropriately defined d-dimensional) percolation threshold, then there is a coefficient of the logarithmic cross sectional area large enough to achieve percolation.

Finally, we study questions concerning the optimal coefficient of logarithmic growth (i.e. the smallest coefficient such that there is percolation). For percolation in $d = 2$, Grimmett proved that such an optimal coefficient exists (Grimmett 1981) and that it is continuous (Grimmett 1983). Moreover, his proof also showed that this coefficient is precisely the dual correlation length of the percolation model on the full lattice. The latter result suggests that the scaling of the optimal coefficient as $p \downarrow p_c$ is related to that of other parameters of the system. Motivated by this, in § 5 we give a different proof of Grimmett's result which shows, as a corollary, that it is possible to construct logarithmic wedges in which the percolation transition is first order in the sense that the percolation probability is discontinuous at the transition point.

2. Definitions and preliminaries

As discussed in § 1, we shall focus attention on three models: the self-avoiding walk, bond percolation and the nearest-neighbour Ising magnet. Each of these models can be defined on general subsets of the bonds of the hypercubic lattice $\mathbb{Z}^d$. The subsets of principal concern to us here will be the telescopes (T, a) defined as follows.

Definition. Let $T = \{T_1, T_2, \ldots, T_j, \ldots\}$ and $a = \{a_1, a_2, \ldots, a_j, \ldots\}$ be sequences of positive integers. The *telescope* (T, a) is the subset of $\mathbb{Z}^d$ with coordinates satisfying

$$\left\{\begin{matrix} 0 \leqslant x_1 \leqslant T_1 \\ |x_2|, |x_3|, \ldots, |x_d| \leqslant a_1 \end{matrix}\right\} \cup \left\{\begin{matrix} T_1 \leqslant x_1 \leqslant T_1 + T_2 \\ |x_2|, |x_3|, \ldots, |x_d| \leqslant a_1 + a_2 \end{matrix}\right\} \cup \ldots$$

$$\cup \left\{\begin{matrix} \sum_{i=1}^{j-1} T_i \leqslant x_i \leqslant \sum_{i=1}^{j} T_i \\ |x_2|, |x_3|, \ldots, |x_d| \leqslant \sum_{i=1} a_i \end{matrix}\right\} \cup \ldots . \tag{2.1}$$

For convenience, we shall often use the notation $A_j = \Sigma_{i=1}^{j} a_i$ for the (strictly increasing) sequence of partial sums.

These telescopes are of course just the increasing symmetric wedges treated by Hammersley and Whittington (1985). In order to avoid excessive provisos, we restrict our analysis to telescopes and hence *symmetric* wedges, since the extension of the results to the non-symmetric case is obvious. It should be noted that every telescope is a *divergent* wedge, since the sequence of widths $A = \{A_1, \ldots, A_j, \ldots\}$ is strictly increasing.

For a general subset of $\mathbb{Z}^d$, the models under consideration have already been defined in § 1, where a few of their properties were reviewed. In particular, we have said that the high temperature phase is characterised by exponential decay of correlations in the sense that

$$\begin{aligned} &\beta < \beta_c^h = \sup\{\beta | \lim_{L\to\infty} (-L^{-1} \log\langle\sigma_0\sigma_L\rangle_\beta) \equiv M(\beta) > 0\} && \text{for the Ising magnet} \\ &p < \pi_c = \sup\{p | \lim_{L\to\infty} (-L^{-1} \log \tau_{0L}(p)) \equiv m(p) > 0\} && \text{for percolation} \\ &\alpha > \alpha_c = \inf\{\alpha | \lim_{L\to\infty} (-L^{-1} \log G_{0L}(\alpha)) \equiv \mu(\alpha) > 0\} && \text{for self-avoiding walks.} \end{aligned} \tag{2.2}$$

(Here L denotes both a point along, say, the x_1 coordinate axis and the distance of the point from the origin.) On the other hand, the system is in the low temperature phase if

$$\begin{aligned} &\beta > \beta_c^l = \inf\{\beta | \langle\sigma_0\rangle_{\beta,+} > 0\} && \text{for the Ising magnet} \\ &p > p_c = \inf\{p | P_\infty(p) > 0\} && \text{for percolation.} \end{aligned} \tag{2.3}$$

Henceforth, we shall use the above notation for the critical points of the full d-dimensional lattice (suppressing the relevant d dependence) and distinguish the corresponding critical points of the telescopes (T, a) with superscripts.

Of course it requires some work to show that the above definitions are meaningful. In particular, the high temperature critical points make sense only if one can establish existence of the masses $M(\beta)$, $m(p)$ and $\mu(\alpha)$. For percolation and the Ising magnet, this is easily accomplished by invoking the Harris-FKG inequalities (Harris 1960, Fortuin *et al* 1971), which provide a subadditive bound. It should be noted that π_c (respectively, β_c^h) is often defined as the point at which the expected cluster size (respectively, the susceptibility) diverges. That the more standard definitions coincide with those given above (on the lattice $\mathbb{Z}^d$) follows from the results of Hammersley (1957) in the case of percolation and from the Simon (1980) inequality for the Ising magnet.

The existence of the mass $\mu(\alpha)$ for self-avoiding walks is somewhat more subtle. The basic problem here is that self-avoiding walks are not subadditive. One way to circumvent this difficulty is to consider *cylinder* self-avoiding walks $w_c: 0 \to L$, which are by definition self-avoiding walks along bonds lying strictly between the hyperplanes $x_1 = 0$ and $x_1 = L$. The corresponding generating function

$$Q_{0L}(\alpha) = \sum_{w_c: 0\to L} \exp(-\alpha|w_c|) \tag{2.4}$$

has a well defined mass $-\lim_{L\to\infty} L^{-1} \log Q_{0L}(\alpha)$. It has been shown (Chayes and Chayes 1986) that this limit is also the decay rate of $G_{0L}(\alpha)$, and hence the latter (which we denote by $\mu(\alpha)$) exists. In § 3, we will use the equivalence of the cylinder and full masses in our proof of the Hammersley and Whittington result.

It is worth noting that one can also define cylinder masses for Bernoulli percolation and the Ising magnet. In the former case, one simply restricts to configurations in which there is a connection between 0 and L within the cylinder region. For the Ising magnet, one must calculate correlations conditioned on free boundary conditions at $x_1 = 0$ and $x_1 = L$. In both cases, equivalence of the cylinder and full mass follows immediately from the Harris-FKG inequality.

The definition of the low temperature critical point also requires some care. For the telescope (T, a), we define the spontaneous magnetisation as $\langle\sigma_0\rangle_{\beta,+}^{(T,a)}$, where $\langle\ \rangle_{\beta,+}^{(T,a)}$ is the limit of the sequence of conditional measures in which the spins along the hyperplane $x_1 = L$ are taken to be $+1$. That a unique limiting measure exists can be established by the usual methods (e.g. by the FKG ordering of the conditional measures).

3. Coincidence of high temperature critical points

In this section, we prove that the high temperature critical point of any telescope coincides with that of the full lattice and show that, in the case of self-avoiding walks, this is equivalent to the result of Hammersley and Whittington (1985).

Our first theorem states that, for all of the models under consideration, the mass in any telescope is equal to the mass of the full d-dimensional lattice for *all* values of parameter. This of course implies coincidence of (high temperature) critical points. The proof—which is a standard technique for establishing continuity of decay rates—will be made explicit only for the case of self-avoiding walks. Identical proofs hold for Bernoulli percolation and the Ising magnet, with the Harris-FKG inequality providing the necessary subadditivity.

Theorem 3.1. Let (T, a) be any telescopic subset of $\mathbb{Z}^d$. Then

$$M^{(T,a)}(\beta) = M(\beta) \qquad \forall\beta\in\mathbb{R}^+$$

$$m^{(T,a)}(p) = m(p) \qquad \forall p\in[0,1]$$

$$\mu^{(T,a)}(\alpha) = \mu(\alpha) \qquad \forall\alpha\in\mathbb{R}^+.$$

Proof. For self-avoiding walks, we rely on the equivalence of the full mass $\mu(\alpha)$ to the cylinder mass, as discussed in § 2. Consider then the cylinder generating function $Q_{0L}(\alpha)$. Indeed, let us begin by further restricting the geometry and considering only those cylinder walks contained in the 'tunnel' $|x_2|, \ldots, |x_d| < A$. The corresponding generating function, $Q_{0L}^A(\alpha)$, has a well defined (finite) decay rate:

$$\mu^A(\alpha) \equiv -\lim_{L\to\infty} L^{-1}\log Q_{0L}^A(\alpha) \tag{3.1}$$

which satisfies

$$\mu^A(\alpha) \geqslant \mu(\alpha) \tag{3.2}$$

and (by subadditivity) provides an *a priori* bound:

$$Q_{0L}^A(\alpha) \leqslant \exp[-\mu^A(\alpha)L]. \tag{3.3}$$

We will demonstrate that $\forall\alpha$

$$\lim_{A\to\infty} \mu^A(\alpha) = \mu(\alpha). \tag{3.4}$$

For simplicity, let us assume $\mu(\alpha) > -\infty$. (The case $\mu(\alpha) = -\infty$ is just as easy to handle.) For any $\varepsilon > 0$, we have

$$Q_{0L}(\alpha) \geqslant \exp[-(\mu(\alpha)-\varepsilon)L] \tag{3.5}$$

once L is large enough. Let $\varepsilon > 0$ and choose L_0 large enough. Using the obvious fact that $Q_{0L_0}(\alpha) = \lim_{A\to\infty} Q^A_{0L_0}(\alpha)$, we have

$$\lim_{A\to\infty} \exp[-\mu^A(\alpha)L_0] \geq \lim_{A\to\infty} Q^A_{0L_0}(\alpha) = Q_{0L_0}(\alpha) \geq \exp[-(\mu(\alpha)-\varepsilon)L_0] \tag{3.6}$$

which completes the demonstration.

Theorem 3.1 is now a foregone conclusion. Indeed, if one is willing to travel a finite distance down any telescope, a region subsuming any tunnel can be found. Explicitly, let $\varepsilon > 0$ and consider a tunnel with A so large that $\mu^A(\alpha) - \varepsilon < \mu(\alpha)$. Then, given the telescope (T, a), let J be the smallest integer such that $A_J \equiv \Sigma_{i=1}^J a_i > A$ and define $R(J) \equiv \Sigma_{i=1}^J T_i$. We have $\forall L$

$$\exp[-\alpha R(J)]Q^A_{0L}(\alpha) \leq Q^{(T,a)}_{0,L+R} \tag{3.7}$$

which implies $\mu(\alpha) = \mu^{(T,a)}(\alpha)$.

It is easily seen that an identical proof holds if $Q_{0L}(\alpha)$ is replaced by $\tau_{0L}(p)$ or $\langle\sigma_0\sigma_L\rangle_\beta$. One need only use the Harris-FKG inequality to obtain the bound (3.3), and to establish monotonicity where necessary.

Defining the high temperature critical points as in equation (2.2), we have the following.

Corollary 3.1.

$$\beta_c^{h(T,a)} = \beta_c^h$$

$$\pi_c^{(T,a)} = \pi_c$$

$$\alpha_c^{(T,a)} = \alpha_c.$$

It is also easy to recover the self-avoiding walk result of Hammersley and Whittington (1984) as follows.

Corollary 3.2. Let λ denote the connectivity constant for self-avoiding walks on $\mathbb{Z}^d$, i.e.

$$\mathcal{N}(n) = \#\{w : 0 \to \cdot \,|\, |w| = n\} \sim \lambda^n$$

(where $\sim$ is meant in the sense of logs and limits) and let $\lambda^{(T,a)}$ denote the corresponding quantity for walks restricted to the telescope $(T, a) \subset \mathbb{Z}^d$. Then

$$\lambda^{(T,a)} = \lambda.$$

Proof. In equation (2.2), α_c was defined as the value of α at which the decay rate $\mu(\alpha)$ becomes non-zero. That e^{α_c} also counts the number of self-avoiding walks (in the sense that $e^{\alpha_c} = \lambda$) can be seen by the following argument (Abraham *et al* 1984, Chayes and Chayes 1985, 1986). We again restrict to cylinder walks and rely on the fact that the connectivity for such walks is identical to the unrestricted connectivity.

First we define $\Gamma_k(L) = \#\{w_c : 0 \to x, x_1 = L \,|\, |w_c| = kL\}$. For a given rational $k \geq 1$, if we let $L \to \infty$ along a sequence of lengths such that $\Gamma_k(L) > 0$, it is not difficult to show that $\Gamma_k(L)$ grows exponentially, i.e.

$$\Gamma_k(L) \sim \exp[\zeta(k)L] \tag{3.8}$$

and, by subadditivity, $\Gamma_k(L) \leqslant \exp[\zeta(k)L] \forall L$. Moreover, $\zeta(k)$ is concave in k, and thus extends to a unique continuous function on all real $k \geqslant 1$. Next, it can be shown that $\zeta(k)$ is related to $\mu(\alpha)$ via a Legendre transform:

$$-\mu(\alpha) = \sup_k \{\zeta(k) - \alpha k\} \tag{3.9}$$

and thus may be identified as the entropy (see Abraham *et al* 1984, Chayes and Chayes 1985, 1986). From (3.9), it is seen that if α_c is defined as the value of α above which $\mu(\alpha) > 0$, then

$$\alpha_c = \sup_k \zeta(k)/k. \tag{3.10}$$

On the other hand, we may count the number of (cylinder) self-avoiding walks. To this end, we write the identity

$$\mathcal{N}(n) = \sum_{\substack{k,L \\ kL=n}} \Gamma_k(L) \tag{3.11}$$

which implies that for any k_0

$$\mathcal{N}(n) \geqslant \Gamma_{k_0}(n/k_0) \tag{3.12}$$

so that

$$\lim_{n\to\infty} n^{-1} \log \mathcal{N}(n) \geqslant \sup_k \zeta(k)/k. \tag{3.13}$$

To obtain a bound in the other direction, note that the number of terms on the RHS of (3.11) cannot exceed n. Thus

$$\mathcal{N}(n) \leqslant n \max_{1 \leqslant k \leqslant n} \Gamma_k(n/k). \tag{3.14}$$

Using the *a priori* bound $\Gamma_k(L) \leqslant \exp[\zeta(k)L]$ and relaxing the restriction on k, this implies

$$n^{-1} \log \mathcal{N}(n) \leqslant n^{-1} \log n + \sup_k \zeta(k)/k \tag{3.15}$$

which, together with (3.13) and (3.10), completes the identification $e^{\alpha_c} = \lambda$.

4. Existence of low temperature and intermediate phases

We now turn to an investigation of the low temperature phase for Bernoulli percolation and the Ising magnet on a certain class of telescopes. Existence of a low temperature phase for Bernoulli percolation in telescopes on $\mathbb{Z}^d$, $d \geqslant 2$, has already been examined by Hammersley and Whittington (1985). They showed that the necessary and sufficient condition for $P_\infty(p) > 0$ at some $p < 1$ is that the cross sectional area of the telescope grow at least logarithmically. In theorem 4.1, we provide a proof of this which holds for both percolation and the Ising magnet. (In the latter case, we show that $\langle\sigma_0\rangle_{\beta,+} > 0$ at some non-trivial β.) Our proof of necessity (theorem 4.1(a)) is essentially the same as that of Hammersley and Whittington (1985), although they did not seem to realise that a proof along these lines holds also for the Ising magnet.

Theorem 4.1, combined with theorem 3.1 on the high temperature critical point, implies the existence of intermediate phases for Bernoulli percolation and the Ising magnet in certain telescopes of logarithmically growing cross sectional area.

Theorem 4.1(a). Suppose there is a $\Delta > 0$ such that the telescope $(T, a) \subset \mathbb{Z}^d$ satisfies

$$\exp[\Delta(2A_j)^{d-1}] \leqslant T_j \tag{4.1}$$

for j large enough. Then percolation does not occur if $p < 1 - e^{-\Delta}$ and the spontaneous magnetisation is zero if $\beta < (1/4)(\Delta - \log 2)$.

Proof. Let us first consider Bernoulli percolation in telescopes satisfying (4.1). By exploiting the well known device of *duality*, the statement that there is no percolation is equivalent to the statement that, with probability one, there is a hypersheet of $(d-1)$-cells dual to vacant bonds which severs the origin from infinity. In telescopes, such sheets must be pierced by the x_1 axis.

We can estimate the probability of such a sheet in the crudest fashion. Indeed, we consider only the event of a completely flat sheet orthogonal to the x_1 axis. Since the events of flat sheets are uncorrelated, the probability that a surface occurs somewhere on the jth segment of the telescope has the lower bound

$$\mathrm{Prob}_p(\exists \text{ a dual hypersheet in the } j\text{th segment}) \geqslant 1 - (1 - (1-p)^{(2A_j)^{d-1}})^{T_j}. \tag{4.2}$$

If $(1-p) > e^{-\Delta}$, it is seen that for j sufficiently large, each segment has a reasonably good chance of exhibiting a sheet. Thus, with probability one, infinitely many separating sheets occur.

The proof for the Ising magnet is almost as easy. Here the object is to show that $\langle \sigma_0 \rangle_{\beta,+}^{(T,a)} = 0$ if $\beta < 0(\Delta)$. To this end, we note that the spontaneous magnetisation must vanish if, with probability one, there is a hypersheet of minus spins which separates the origin from infinity.

As in the percolation proof, the probability of this can be bounded below by the probability of a flat sheet of minus spins orthogonal to the x_1 axis. Unfortunately, these events are correlated between layers. Let us therefore bound the probability that a flat minus sheet occurs in any layer by the probability that it occurs in an odd layer. By the FKG inequality, this is in turn bounded below by the probability of a flat minus sheet in an odd layer conditioned on the event that every even layer has exclusively plus spins. Estimating this, we obtain the bound

$$\mathrm{Prob}(\exists \text{ a minus hypersheet in the } j\text{th segment})$$
$$\geqslant 1 - (1 - (\tfrac{1}{2} e^{-4\beta})^{(2A_j)^{d-1}})^{(T_j - 1)/2} \tag{4.3}$$

from which the result follows.

A converse statement, which demonstrates the existence of a low temperature phase, is also straightforward.

Theorem 4.1(b). Suppose there is a $\Delta < \infty$ such that the telescope $(T, a) \subset \mathbb{Z}^d$ satisfies

$$T_j \leqslant \exp[\Delta(2A_j)^{d-1}] \tag{4.4}$$

for all j large enough. Then percolation occurs if p is sufficiently close to one and the spontaneous magnetisation is positive for β sufficiently large.

Proof. Consider first the Ising magnet, with conditional measures $\langle\ \rangle_{\beta,+}^{(T,a)_L}$ obtained by setting to plus all spins along the hyperplane $x_1 = L$. We must show that, uniformly in L, the origin retains spontaneous magnetisation for β large enough. To this end, we use the Peierls-Griffiths estimate (Peierls 1936, Griffiths 1964):

$$\langle \sigma_0 \rangle_{\beta,+}^{(T,a)_L} \geqslant 1 - 2 \sum_{\gamma \in \Gamma_L(T,a)} \mathrm{Prob}(\gamma). \tag{4.5}$$

Here $\Gamma_L(T, a)$ is the set of all contours, γ, composed of dual $(d-1)$-cells which separate spins of opposite type and sever the origin from the boundary $x_1 = L$ of the telescope $(T, a)_L$. For any given contour $\gamma \in \Gamma_L(T, a)$, $\text{Prob}(\gamma) \leq \exp(-2\beta|\gamma|)$, where $|\gamma|$ denotes the volume (i.e. number of $(d-1)$-cells) of the contour γ.

In order to obtain an estimate on the sum in (4.5), we must classify all contours in $\Gamma_L(T, a)$. A natural classification scheme is in terms of the earliest segment of the telescope which the boundary, $\partial\gamma$, of the contour γ visits. We will say that $\gamma \in \Gamma^{(j)}(T, a)$ if $\Sigma_{i=1}^{j-1} T_i \leq \min(x_1 \text{ coordinate of } \partial\gamma) < \Sigma_{i=1}^{j} T_i$. The $\Gamma^{(j)}(T, a)$ clearly form a (disjoint) partition of $\Gamma_L(T, a)$. It will suffice to obtain an upper bound on the number of contours $\gamma \in \Gamma^{(j)}(T, a)$ and a lower bound on the area of each such contour.

For a given contour γ, let us denote by $\{p_\gamma\}$ the set of $(d-2)$-cells of $\partial\gamma$ with minimum x_1 coordinate. We will overestimate the size of $\Gamma^{(j)}(T, a)$ by calculating the probability of a contour with $p_\gamma = p$ for some fixed p in the boundary of the jth segment (thus double-counting any contour for which $\{p_\gamma\}$ contains more than a single $(d-2)$-cell).

It turns out that we must divide each set $\Gamma^{(j)}(T, a)$ into two sets depending on whether p_γ has the x_1 coordinate in the initial 'vertical' annular boundary of the jth segment or whether p_γ lies along the 'horizontal' boundary of the jth segment (see figure 1). Let us denote the former set by $\Gamma_{\mathrm{v}}^{(j)}(T, a)$ and the latter by $\Gamma_{\mathrm{h}}^{(j)}(T, a)$.

It is reasonable to expect that the probability of a contour in the first class, $\Gamma_{\mathrm{v}}^{(j)}(T, a)$, should be bounded above, independent of T_j. Let us show that this is the case. To this end, note that any contour $\gamma \in \Gamma_{\mathrm{v}}^{(j)}(T, a)$ has its 'leftmost' boundary cell p_γ on the initial boundary of the jth segment, which is a square annulus of internal radius A_{j-1} and external radius A_j. Let us denote by N_γ the minimum distance of p_γ from the internal hole of the annulus. We can further partition $\Gamma_{\mathrm{v}}^{(j)}(T, a)$ into sets $\Gamma_{\mathrm{v},N}^{(j)}(T, a)$ in which $N_\gamma = N$, $N = 0, \ldots, N_{\max}^{(j)}$, where $N_{\max}^{(j)}$ is a function of A_{j-1} and A_j. The volume of any $\gamma \in \Gamma_{\mathrm{v},N}^{(j)}(T, a)$ has the obvious lower bound $|\gamma| \geq (2A_{j-1})^{d-1} + N$. Now let us estimate the number of such contours. To do this, first note that the number of $(d-2)$-cells, p, located a distance N from the internal hole is $2(d-1)[2(A_{j-1} + N)]^{d-2}$. For any given p, we use the elementary fact that the number of contours of volume n with boundary passing through a fixed $(d-2)$-cell is bounded above by a Peierls estimate $\exp(\mu_d n)$, $\mu_d < \infty$. Thus

$$\begin{aligned}\text{Prob}(\gamma \in \Gamma_{\mathrm{v},N}^{(j)}(T, a)) &\leq 2(d-1)[2(A_{j-1} + N)]^{d-2} \sum_{n \geq (2A_{j-1})^{d-1} + N} \exp(-2\beta n) \exp(\mu_d n) \\ &\leq C_{\mathrm{v}}(\beta)(2A_{j-1})^{d-2} N^{d-2} \exp\{-\kappa(\beta)[(2A_{j-1})^{d-1} + N]\} \qquad (4.6)\end{aligned}$$

with $C_{\mathrm{v}}(\beta) \to 1$ and $\kappa(\beta) \to \infty$ as $\beta \to \infty$. Finally, we may relax the constraint that

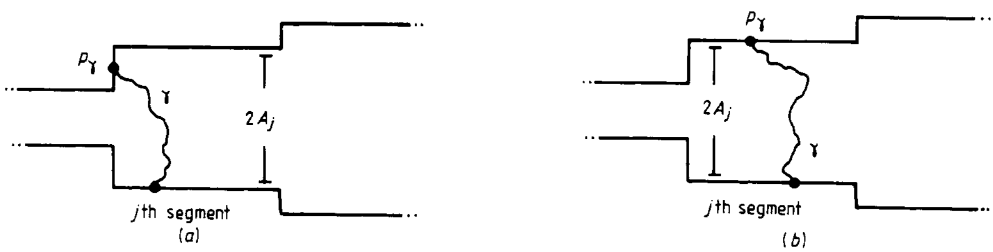

Figure 1. Contours γ; (a) $\gamma \in \Gamma_{\mathrm{v}}^{(j)}(T, a)$; (b) $\gamma \in \Gamma_{\mathrm{h}}^{(j)}(T, a)$.

$N \leqslant N_{\max}^{(j)}$ and sum over all N to obtain

$$\begin{aligned}\mathrm{Prob}(\gamma \in \Gamma_{\mathrm{v}}^{(j)}(T, a)) &\leqslant C_{\mathrm{v}}(\beta)(2A_{j-1})^{d-2} \\ &\quad \times \exp[-\kappa(\beta)(2A_{j-1})^{d-1}] \sum_{N=0}^{\infty} N^{d-2} \exp[-\kappa(\beta)N] \\ &= C'_{\mathrm{v}}(\beta)(2A_{j-1})^{d-2} \exp[-\kappa(\beta)(2A_{j-1})^{d-1}] \qquad (4.7)\end{aligned}$$

which is independent of T_j, as expected.

The second (and principal) class of contours, $\Gamma_{\mathrm{h}}^{(j)}(T, a)$, is, in fact, easier to handle. Here we see that each $\gamma \in \Gamma_{\mathrm{h}}^{(j)}(T, a)$ has volume $|\gamma| \geqslant (2A_j)^{d-1}$. In order to count the number of contours, we first note that the number of $(d-2)$-cells, p, in the 'horizontal' boundary of the jth segment is $2(d-1)T_j(2A_j)^{d-2}$, and then use the Peierls estimate on the number of contours passing through a given p. The result is

$$\begin{aligned}\mathrm{Prob}(\gamma \in \Gamma_{\mathrm{h}}^{(j)}(T, a)) &\leqslant 2(d-1)T_j(2A_j)^{d-2} \sum_{n \geqslant (2A_j)^{d-1}} \exp(-2\beta n) \exp(\mu_d n) \\ &\leqslant C_{\mathrm{h}}(\beta) T_j (2A_j)^{d-2} \exp[-\kappa(\beta)(2A_j)^{d-1}]. \qquad (4.8)\end{aligned}$$

Finally, we use our assumption (4.4) on the growth rate of the telescope to obtain

$$\mathrm{Prob}(\gamma \in \Gamma_{\mathrm{h}}^{(j)}(T, a)) \leqslant C_{\mathrm{h}}(\beta)(2A_j)^{d-2} \exp\{-[\kappa(\beta) - \Delta](2A_j)^{d-1}\}. \qquad (4.9)$$

Summing (4.7) and (4.9) over all j, and recalling that $\kappa(\beta) \to \infty$ as $\beta \to \infty$, we see that $\Sigma_{\gamma \in \Gamma_L(T,a)} \mathrm{Prob}(\gamma)$ tends to zero uniformly in L as $\beta \to \infty$. Thus, by taking β large enough, (4.5) implies that $\langle \sigma_0 \rangle_{\beta,+}^{(T,a)_L} > 0$ uniformly in L.

The proof for percolation is essentially identical. We simply use the analogue of (4.5) for the probability that the origin is connected to the plane $x_1 = L$ by a path of occupied bonds within the telescope $(T, a)_L$. Then the contours γ are composed of $(d-1)$-cells dual to vacant bonds, so that the factor of $\exp(-2\beta)$ in the estimates (4.6) and (4.8) is replaced by $(1-p)$.

Remark. It turns out that *convergence* of the sum $\Sigma_{\gamma \in \Gamma_L(T,a)} \mathrm{Prob}(\gamma)$, uniformly in L, is sufficient to guarantee $\langle \sigma_0 \rangle_{\beta,+}^{(T,a)} > 0$ or $P_\infty^{(T,a)}(p) > 0$. Indeed, by the Borel-Cantelli lemma, finiteness of the sum implies that, with probability one, only a finite number of contours appear in the system. From this, one can easily show that, with positive probability, there are *no* contours.

Corollary 4.1. If (T, a) satisfies the conditions of theorem 4.1(b), then there is a low temperature phase, i.e.

$$\beta_{\mathrm{c}}^{\mathrm{I}(T,a)} \leqslant \tfrac{1}{2}(\mu_d + \Delta) < \infty$$

$$p_{\mathrm{c}}^{(T,a)} \leqslant 1 - \exp[-(\mu_d + \Delta)] < 1.$$

Furthermore, combining theorems 3.1, 4.1(a) and 4.1(b), we have the following.

Corollary 4.2. Suppose that (T, a) satisfies

$$\lim_{j \to \infty} \frac{\log T_j}{(2A_j)^{d-1}} = \Delta < \infty.$$

Then $\beta_c^{l(T,a)} \geqslant (1/4)(\Delta - \log 2)$ and $p_c^{(T,a)} \geqslant 1 - e^{-\Delta}$. In particular, if $(1/4)(\Delta - \log 2) > \beta_c^{h(T,a)}$ or $1 - e^{-\Delta} > \pi_c^{(T,a)}$, there is an intermediate phase characterised by subexponential decay of correlations but zero spontaneous magnetisation or percolation probability.

5. The low temperature critical point and first order transitions in percolation

In the last section, we proved that there is a low temperature phase in logarithmic telescopes, but our estimates on the critical point were far from optimal. Here, we restrict our attention to percolation and show that the critical point in wedges of logarithmically growing cross sectional area may be made arbitrarily close to that of the full lattice in the sense that if p exceeds a critical value $\hat{p}_c^\infty$, introduced by Aizenman *et al* (1983) and conjectured to equal p_c of the full lattice $\mathbb{Z}^d$, $d > 2$, it is possible to find a coefficient of logarithmic growth sufficiently large that there is percolation in the corresponding wedge. Assuming $\hat{p}_c^\infty = p_c$, this generalises the two-dimensional result of Grimmett (1981, 1983) and proves the conjecture of Hammersley and Whittington (1985).

In the second part of this section, we restrict attention further to two-dimensional percolation and show that there exists a special class of logarithmically growing wedges for which the percolation transition is discontinuous.

5.1. *The low temperature critical point for wedges on $\mathbb{Z}^d$, $d > 2$*

Consider the bond percolation model on quadrant slabs $\hat{L}_k = (\mathbb{Z}^+)^2 \times \{0, \dots, k\}^{d-2}$, $d > 2$, where $(\mathbb{Z}^+)^2$ denotes the positive quadrant of $\mathbb{Z}^2$. One may define a corresponding percolation threshold $\hat{p}_c^k$ as the value of p above which the origin has positive probability of being connected to infinity within the slab $\hat{L}_k$. The critical point $\hat{p}_c^\infty$ is then defined as the limit of these slab thresholds:

$$\hat{p}_c^\infty \equiv \lim_{k \to \infty} \hat{p}_c^k. \tag{5.1}$$

It is natural to conjecture, as was done in Aizenman *et al* (1983), that $\hat{p}_c^\infty = p_c$, where p_c is the percolation threshold of the full lattice $\mathbb{Z}^d$. The content of theorem 5.1 (below) is that the threshold of a logarithmic wedge may also be made arbitrarily close to $\hat{p}_c^\infty$ by taking the coefficient of the logarithm large enough.

Our proof of theorem 5.1 relies on some rescaling techniques, introduced in Aizenman *et al* (1983) and briefly reviewed below. There it was shown that, from the lattice $\mathbb{Z}^d$, one can construct a rescaled lattice with squares of size $J \times J \times k^{d-2}$ as 'sites' and a nearest-neighbour distance of L ($L > J, k$). Moreover, for each pair of these nearest-neighbour sites, it is possible to define a renormalised bond event with the following properties: (i) the bond events are transitive in the sense that if one occurs between sites i and j, and another occurs between sites j and l, then i must be connected to l by a path of (microscopic) occupied bonds; (ii) a given bond event is independent of all other bond events except those with which it shares an endpoint site and (iii) if $p > \hat{p}_c^k$, the probability of a given bond event tends to one as J and L tend to infinity in an appropriate manner. Thus, provided that one considers bond events on $2d$ independent rescaled sublattices, one recovers a Bernoulli bond system with an effective bond density $\rho(J, L)$ that can be made arbitrarily close to one by taking J and L large enough.

Theorem 5.1. Let $d>2$ and suppose $p>\hat{p}_c^{\infty}$. Then there is a $\Delta=\Delta(p)>0$ such that any telescope $(T, a)\subset\mathbb{Z}^d$ satisfying

$$T_j\leqslant\exp[\Delta(p)(2A_j)^{d-1}] \tag{5.2}$$

has $p>p_c^{(T,a)}$.

Proof. The proof is a combination of the Peierls argument of theorem 4.1(b) and the rescaling construction reviewed above. Let $\Delta_0<\infty$. Then, by corollary 4.1 to theorem 4.1(b), if $(1-p)\exp[(\mu_d+\Delta_0)]<1$, there is percolation in any telescope which satisfies $T_j\leqslant\exp[\Delta_0(2A_j)^{d-1}]$ for j large enough. Indeed, if we require that p satisfy the stronger condition $(1-p)\exp[(2d\mu_d+\Delta_0)]<1$, then the Peierls estimate may be performed on independent sublattices with the same result.

Next, we use the rescaling construction to show that for any $p>\hat{p}_c^{\infty}$, there is a rescaled lattice for which the above analysis holds. To this end, we observe that if $p>\hat{p}_c^{\infty}$, then there is a $k<\infty$ such that $p>\hat{p}_c^{k}$. Thus, for fixed $\Delta_0<\infty$, we can find $J(=J(p))$ and $L(=L(p))$ large enough so that the effective bond density of the rescaled lattice satisfies

$$[1-\rho(J, L)]\exp[(2d\mu_d+\Delta_0)]<1. \tag{5.3}$$

Now define

$$\Delta(p)\equiv\tfrac{1}{2}\Delta_0/L(p)^{d-1}>0 \tag{5.4}$$

and consider any telescope (T, a) which satisfies

$$T_j\leqslant\exp[\Delta(p)(2A_j)^{d-1}]. \tag{5.5}$$

Rescaling the bonds of (T, a), we obtain a telescope $(\bar{T}, \bar{a})$ in which the effective length of the jth segment is $\bar{T}_j=[T_j/L]_{\mathrm{I}}$ and the effective cross section is $(2\bar{A}_j)^{d-1}=[2A_j/L]_{\mathrm{I}}^{d-1}$. (Here $[c]_{\mathrm{I}}$ denotes the largest integer smaller than c.) Using (5.4) and (5.5), we have

$$\bar{T}_j\leqslant\exp[\Delta_0(2\bar{A}_j)^{d-1}] \tag{5.6}$$

for j large enough. However, by (5.3) and the Peierls argument, this implies that the rescaled bonds percolate on the telescope $(\bar{T}, \bar{a})$, and hence that the original bonds percolate on (T, a).

5.2. Characterisation of the percolation transition in wedges on $\mathbb{Z}^2$

Thus far, we have examined the question of whether a transition occurs in wedges and, if so, the location of the critical point. Here, we restrict attention to wedges of $\mathbb{Z}^2$ and examine the nature of the percolation transition. The key ingredient is a new proof of Grimmett's result (1981, 1983), which says that the optimal coefficient of logarithmic growth is exactly the dual correlation length of the percolation model on the full lattice $\mathbb{Z}^2$. Equivalently, this means that the optimal coefficient of exponential growth of the T_j is the dual mass for two-dimensional percolation. The nature of the transition will then be determined by corrections to exponential growth of the T_j. For example, we show that whenever these corrections are summable in j, the percolation probability is discontinuous at $p_c^{(T,a)}$. This is in marked contrast to the second order nature of the percolation transition on the full lattice $\mathbb{Z}^2$, for which $P_\infty(p)$ is known to be continuous (Russo 1978).

Consider a telescope $(T, a) \subset \mathbb{Z}^2$. In order to avoid excessive provisos, we shall henceforth assume that $\lim_{j\to\infty}(2A_j)^{-1} \log T_j \equiv \Delta$ exists. For any such telescope (with $\Delta > 0$), Grimmett (1983) found that there is a critical value $p(\Delta)$ above which percolation occurs. Indeed, he showed that $p(\Delta)$ is continuous and strictly monotone.

An interesting consequence of Grimmett's proof, and the one with which we are principally concerned, is that $p(\Delta)$ is related to more familiar quantities in percolation theory. The relationship is most natural within the context of the inverse function $\Delta : (p_c, 1) \to (0, \infty]$ defined as

$$\Delta(p) = \sup\{\Delta | \lim_{j\to\infty} (2A_j)^{-1} \log T_j < \Delta \text{ and } p > p_c^{(T,a)}\}. \tag{5.7}$$

In other words, $\Delta(p)$ is the largest coefficient of exponential growth for which there is percolation in the telescope at density greater than p. In theorem 5.2, we give a new proof of Grimmett's theorem (1981, 1983) relating $\Delta(p)$ to the decay rate of the dual connectivity function, defined below.

Consider Bernoulli bond percolation on $\mathbb{Z}^2$ at density p. Denote by $\tau_{0L}^*(p)$ the probability that the origin of the dual lattice $\mathbb{Z}^{2*}$ is in the same connected cluster of *dual* bonds as the point $(L+\frac{1}{2}, \frac{1}{2}) \in \mathbb{Z}^{2*}$. By subadditivity, it is easy to show that

$$\lim_{L\to\infty} (-L^{-1} \log \tau_{0L}^*(p)) \equiv m^*(p) \tag{5.8}$$

exists. Furthermore, it follows from the results of Hammersley (1957), Russo (1978) and Seymour and Welsh (1978) that $m^*(p) > 0$ whenever $p > p_c$.

For the proof of theorem 5.2, we will need two other functions closely related to $\tau_{0L}^*(p)$. First, denote by $t_{0L}^*(p)$ the probability that the dual origin is connected (by dual bonds) to *any* point on the plane $x_1 = L + \frac{1}{2}$. Second, denote by $R_L^*(p)$ the probability that there is a left-right dual crossing of an $L \times L$ square. It is not hard to show (Chayes *et al* 1985 or Chayes and Chayes 1986) that for $p > p_c$, $t_{0L}^*(p)$ and $R_L^*(p)$ also decay exponentially in the sense of (5.8) with decay rate $m^*(p)$. Finally, we note that t_{0L}^* obeys the subadditive bound $t_{0L}^* \leq e^{-m^* L} \forall L$.

For bond percolation on $\mathbb{Z}^2$, $m^*(p) = m(1-p)$ by self-duality. It should be noted, however, that this plays no role in either the properties reviewed above or our proof of theorem 5.2.

Theorem 5.2. Whenever $p > p_c$,

$$\Delta(p) = m^*(p).$$

Proof. Let $p > p_c$ and consider a telescope $(T, a) \subset \mathbb{Z}^2$ which satisfies $\lim_{j\to\infty}(2A_j)^{-1} \log T_j = \Delta$. We must show that there is no percolation in (T, a) if $\Delta > m^*(p)$, while there is percolation if $\Delta < m^*(p)$. This will be done by Peierls' arguments similar to those of theorem 4.1.

First suppose that $\Delta > m^*$. Let us divide the jth segment into $n_j \equiv [T_j/2A_j]_I$ disjoint $2A_j \times 2A_j$ squares. The probability of a dual surface in the jth segment is bounded below by the probability of such a surface occurring in at least one of these squares. This is given by $1 - (1 - R_{2A_j}^*)^{n_j}$, which tends to one since $\Delta > m^*$. Thus there is no percolation.

Now suppose $\Delta < m^*$. As in the proof of theorem 4.1(b), let us estimate the probability of dual contours by decomposing contours into the sets $\Gamma_v^{(j)}(T, a)$ and $\Gamma_h^{(j)}(T, a)$. Consider first contours in $\Gamma_h^{(j)}(T, a)$. As before, the number of starting points for such contours is $2T_j$. Now, however, we use the fact that the probability of a contour from a given point on the boundary of the jth segment to some point of the boundary of the $(j+k)$th segment, $k=0, 1, \ldots$, is bounded above by $t^*_{0,2A_j}$. Using the subadditive bound on $t^*_{0,2A_j}$, we have

$$\mathrm{Prob}(\gamma \in \Gamma^{(j)}_{\mathrm{h}}(T, a)) \leq 2T_j \exp[-m^*(p)(2A_j)] \tag{5.9}$$

which is the analogue of (4.8).

The contours in $\Gamma^{(j)}_{\mathrm{v}}(T, a)$ are also easy to handle. Again, we partition $\Gamma^{(j)}_{\mathrm{v}}(T, a)$ into the sets $\Gamma^{(j)}_{\mathrm{v},N}(T, a)$, depending on the minimum distance of the starting point of the contour to the internal hole of the annulus. In two dimensions, there are only two starting points for contours in a given $\Gamma^{(j)}_{\mathrm{v},N}(T, a)$. For each of these points, we bound the probability of a contour by $t_{0,2A_{j-1}+N} \leq \exp[-m^*(p)(2A_{j-1}+N)]$. Summing over N, we obtain an analogue of (4.7):

$$\mathrm{Prob}(\gamma \in \Gamma^{(j)}_{\mathrm{v}}(T, a)) \leq C_{\mathrm{v}}(p) \exp[-m^*(p)(2A_{j-1})]. \tag{5.10}$$

Summing (5.9) and (5.10) over all j, and using the fact that $m^* > \Delta$, we see that

$$\sum_{\gamma \in \Gamma_L(T,a)} \mathrm{Prob}(\gamma) < \infty \tag{5.11}$$

uniformly in L. By the Borel-Cantelli lemma, etc, this implies that, with positive probability, no dual contours occur. Evidently, the percolation probability is positive.

Corollary 5.1. Let $p_0 > p_c$ and suppose

$$T_j = f(A_j) \exp[m^*(p_0)(2A_j)]. \tag{5.12}$$

If $\Sigma_j f(A_j) < \infty$, then the percolation probability $P^{(T,a)}_\infty(p)$ is discontinuous at $p_0 = p^{(T,a)}_c$.

Proof. That $p^{(T,a)}_c = p_0$ (and hence that $P^{(T,a)}_\infty(p) = 0$ whenever $p < p_0$) follows immediately from the theorem. To prove that $P^{(T,a)}_\infty(p_0) > 0$, note that the condition $\Sigma_j f(A_j) < \infty$ implies that (5.9) is summable in j at $p = p_0$. Since (5.10) is summable for every $p > p_c$, we see that (5.11) holds also at $p = p_0$, and hence that the percolation probability is positive.

Remark. It is clear that, by suitable adjustment of the correction, $f(A_j)$, to exponential growth, it is possible to produce a variety of critical behaviours, of which the first order transition demonstrated above is only one example.

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